

# M208

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## TMA 05

## 2019J

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(Covers Book E)

Cut-off date 12 March 2020

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You can submit your TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on the TMA, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' tab of the M208 website.

In the wording of the questions:

- *write down, list* or *state* means 'write down without justification' (unless otherwise stated)
- *find, determine, calculate, derive, evaluate* or *solve* means 'show all your working'
- *prove, show, deduce* or *verify* means 'justify each step'
- *sketch* means 'sketch without justification' and *describe* means 'describe without justification' (both unless otherwise stated).

In particular, if you use a definition, result or theorem to go from one line to the next, then you must state clearly which fact you are using – for example, you could quote the relevant unit and page, or give a Handbook reference. Remember that when you use a theorem, you must demonstrate that all the conditions of the theorem are satisfied.

The number of marks assigned to each part of a question is given in the right-hand margin, to give you a rough indication of the amount of time that you should spend on each part.

Your work should be written in a good mathematical style, as demonstrated by the exercise and worked exercise solutions in the study texts. You should explain your solutions carefully, using appropriate notation and terminology, defining any symbols that you introduce, and writing in proper sentences. Five marks (referred to as good mathematical communication, or GMC, marks) on this TMA are allocated for how well you do this.

Your score out of 5 for GMC will be recorded against Question 7. (You do not have to submit any work for this particular question.)

You should read the information on the front page of this booklet before you start working on the questions.

**Question 1** (Unit E1) – 13 marks

Show that each of the following sets is a group under matrix multiplication.

(a)  $X = \left\{ \begin{pmatrix} a & 0 \\ b & 1/a \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$  [8]

(b)  $Y = \{\mathbf{I}, \mathbf{P}, \mathbf{Q}, \mathbf{R}\}$ , where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad [5]$$

**Question 2** (Units E1 and E2) – 13 marks

The group table of a group  $G$  is shown below.

	$e$	$p$	$q$	$r$	$s$	$t$	$u$	$v$	$w$	$x$	$y$	$z$
$e$	$e$	$p$	$q$	$r$	$s$	$t$	$u$	$v$	$w$	$x$	$y$	$z$
$p$	$p$	$q$	$e$	$y$	$u$	$w$	$z$	$r$	$x$	$t$	$v$	$s$
$q$	$q$	$e$	$p$	$v$	$z$	$x$	$s$	$y$	$t$	$w$	$r$	$u$
$r$	$r$	$z$	$t$	$s$	$e$	$y$	$v$	$x$	$p$	$u$	$q$	$w$
$s$	$s$	$w$	$y$	$e$	$r$	$q$	$x$	$u$	$z$	$v$	$t$	$p$
$t$	$t$	$r$	$z$	$x$	$w$	$u$	$e$	$q$	$y$	$p$	$s$	$v$
$u$	$u$	$x$	$v$	$p$	$y$	$e$	$t$	$z$	$s$	$r$	$w$	$q$
$v$	$v$	$u$	$x$	$z$	$q$	$r$	$y$	$w$	$e$	$s$	$p$	$t$
$w$	$w$	$y$	$s$	$t$	$x$	$z$	$p$	$e$	$v$	$q$	$u$	$r$
$x$	$x$	$v$	$u$	$w$	$t$	$s$	$q$	$p$	$r$	$e$	$z$	$y$
$y$	$y$	$s$	$w$	$u$	$p$	$v$	$r$	$t$	$q$	$z$	$e$	$x$
$z$	$z$	$t$	$r$	$q$	$v$	$p$	$w$	$s$	$u$	$y$	$x$	$e$

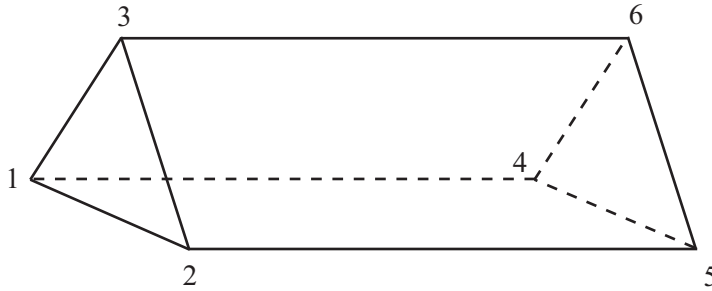
The subsets  $H = \{e, x, y, z\}$  and  $K = \{e, p, q\}$  are subgroups of  $G$ .

(a) For each of  $H$  and  $K$ , list all of its distinct left cosets and all of its distinct right cosets. [8]

(b) One of the subgroups  $H$  and  $K$  is a normal subgroup of  $G$ . State which subgroup this is, and justify your answer. For this subgroup, construct the group table of its quotient group in  $G$ , and write down a standard group listed in the Handbook to which this quotient group is isomorphic. [5]

**Question 3** (Unit E2) – 24 marks

The solid  $P$  shown below is a prism with three identical rectangular faces and an equilateral triangle at each end. The locations of its vertices have been labelled so that the symmetries of the prism can be represented as permutations of the set  $\{1, 2, 3, 4, 5, 6\}$ .



- (a) Let  $r$  be the rotation anticlockwise through  $2\pi/3$  about the horizontal axis of symmetry, looking from left to right, and let  $x$  be the reflection in the vertical plane through the vertices at locations 3 and 6. Write down these symmetries in cycle form as permutations of the set  $\{1, 2, 3, 4, 5, 6\}$ . [2]
- (b) The permutation  $t = (1\ 4)(2\ 5)(3\ 6)$  is an element of  $S(P)$ , the symmetry group of the prism  $P$ . Describe it geometrically. [1]
- (c) Write down all the symmetries of the prism in cycle form as permutations of the set  $\{1, 2, 3, 4, 5, 6\}$ . [9]
- (d) Determine, as permutations in cycle form, the conjugates  $r \circ x \circ r^{-1}$  and  $x \circ t \circ x^{-1}$ . [2]
- (e) Write down all the conjugacy classes of the group  $S(P)$ , briefly explaining your reasoning. [8]
- (f) Prove that the cyclic subgroup  $\langle r \rangle$  is a normal subgroup of  $S(P)$ . [2]

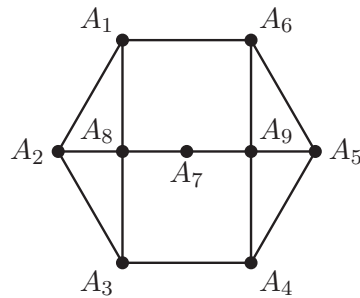
**Question 4** (Unit E3) – 15 marks

This question concerns the groups  $(\mathbb{C}, +)$  and  $(\mathbb{C}^*, \times)$ , where  $\mathbb{C}$  is the set of all complex numbers and  $\mathbb{C}^*$  is the set of all non-zero complex numbers.

- (a) For each of the following mappings, determine whether or not it is a homomorphism, justifying your answer.
  - (i)  $\phi_1 : (\mathbb{C}, +) \longrightarrow (\mathbb{C}, +)$   
 $z \longmapsto |z|$
  - (ii)  $\phi_2 : (\mathbb{C}, +) \longrightarrow (\mathbb{C}, +)$   
 $z \longmapsto z - 6iz$
  - (iii)  $\phi_3 : (\mathbb{C}^*, \times) \longrightarrow (\mathbb{C}, +)$   
 $z \longmapsto \frac{3-z}{z}$  [10]
- (b) For each mapping  $\phi$  in part (a) that is a homomorphism, determine  $\text{Ker } \phi$  and  $\text{Im } \phi$ , and write down a standard group listed in the Handbook that is isomorphic to the quotient group  $G/\text{Ker } \phi$ , where  $G$  is the group that is the domain of the homomorphism. [5]

**Question 5** (Unit E4) – 15 marks

The figure shown below is a regular hexagon with vertices  $A_1, A_2, A_3, A_4, A_5$  and  $A_6$ , and centre  $A_7$ . Three pairs of vertices have been joined to form horizontal and vertical lines, and these lines intersect at points  $A_8$  and  $A_9$  as shown.



The symmetry group of the figure is  $G = \{e, a, r, s\}$ , where  $e$  is the identity,  $a$  is the rotation through  $\pi$  about the centre,  $r$  is the reflection in the vertical axis of symmetry, and  $s$  is the reflection in the horizontal axis of symmetry. An action of  $G$  on the set

$$X = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$$

is defined by

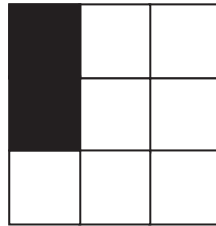
$$g \cdot P = g(P)$$

for all  $g \in G$  and all  $P \in X$ . (This is a group action by Theorem E59.)

- (a) Write down all the orbits of the action of  $G$  on  $X$ . [6]
- (b) Write down the stabiliser of each of the points  $A_1, A_2$  and  $A_7$ . [3]
- (c) Write down the fixed set of the symmetry  $s$ . [2]
- (d) Is there a subgroup of  $G$  that is not the stabiliser of any element of  $X$ ? Justify your answer. [4]

**Question 6** (Unit E4) – 15 marks

An ornament is made of nine square pieces of coloured glass, all the same size, arranged to make a larger square. The middle square is coloured white, and the other eight squares can be coloured either black or white. One such ornament is shown below.



Use the Counting Theorem to calculate how many different ornaments can be made if two ornaments are regarded as being the same when a rotation or reflection takes one to the other.

[15]

**Question 7** (Book E) – 5 marks

Five marks on this assignment are allocated for good mathematical communication in your answers to Questions 1 to 6.

You do not have to submit any extra work for Question 7, but you should check through your assignment carefully, making sure that you have explained your reasoning clearly, used notation correctly and written in proper sentences.

[5]